Correlation Functions of Winding Strings in AdS$_3$ [1]

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Abstract. We review certain results for amplitudes of spectral flowed operators in string theory on AdS$_3$. We present the modified Knizhnik-Zamolodchikov and null vector equations to be satisfied by correlators including $w = 1$ operators. We then discuss the three point function of two $w = 1$ and one $w = 0$ operators in the x-basis, and perform a consistency check on the definition of the $w = 1$ operator. We finally exhibit the steps in the calculation of the winding conserving four point functions for operators in arbitrary spectral flowed sectors, both in the m- and x-basis, under the only assumption that at least one of the operators is in the spectral flowed image of the highest weight discrete representation.

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1. INTRODUCTION

In this article we consider certain correlation functions in the $SL(2,R)$ WZW model, which describes strings propagating in the three dimensional Anti-de Sitter space (AdS$_3$). The motivations are that to study such a model would allow to investigate aspects of the AdS/CFT correspondence beyond the supergravity approximation, and that the propagation of strings in this non-compact background is not well-understood yet. The applications of this model to black hole physics come as an additional motivation. However, diverse difficulties arise due to the non-rational structure of this CFT.

Here we employ two different basis to calculate the amplitudes of the theory. In the x-basis the isospin parameter is understood as the coordinate of the boundary of the AdS$_3$ space, whereas in the m-basis the generators of the global isometry are diagonalized. The transformation relating an operator $\Phi_{j;m,\bar{m}}(z,\overline{z})$ in the m-basis to a corresponding operator $\Phi_j(x,\overline{x})$ in the x-basis is given by

$$\Phi_{j;m,\bar{m}}(z,\overline{z}) = \int \frac{d^2x}{|x|^2}\, x^{j-m}\, \overline{x}^{j-\bar{m}}\, \Phi_j(x,\overline{x};z,\overline{z}) \; . \tag{1}$$

The states in the long strings sectors correspond to continuous representations of $SL(2,R)$ with spin $j = \frac{1}{2} + is$, $s \in \mathbf{R}$, whereas short string sectors are discrete representations with spin $j \in \mathbf{R}$ and unitarity bound $\frac{1}{2} < j < \frac{k-1}{2}$. The conformal weight of the primary fields is given by $\Delta_j = -\frac{j(j-1)}{k-2}$, where $k > 2$.

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[1] Based on work in collaboration with E. Herscovich and C. Núñez [1, 2].

There is however an additional feature of the $SL(2,R)$ CFT which is the fact that it contains different sectors which are labeled by the winding number or spectral flow parameter $w \in \mathbf{Z}$.[2] Thus we will also consider here spectral flowed operators which we write $\Phi^{w,j}_{J,M;\bar{J},\bar{M}}(z)$ in the m-basis, or $\Phi^{w,j}_{J,\bar{J}}(x,z)$ in the x-basis. The conformal weight and spin are given by

$$\Delta^w_j = -\frac{j(j-1)}{k-2} - mw - \frac{k}{4}\,w^2 \;, \tag{2}$$

with a similar expression for $\bar{\Delta}^w_j$, and

$$J = |\,M\,| = |\,m + \frac{kw}{2}\,|\,, \qquad\qquad \bar{J} = |\,\bar{M}\,| = |\,\bar{m} + \frac{kw}{2}\,|\;\;. \tag{3}$$

In particular, a $w = 1$ operator in the x-basis is obtained from a $w = 0$ operator through [3]

$$\Phi^{w=1,j}_{J,\bar{J}}(x,\bar{x};z,\bar{z}) \equiv \lim_{\varepsilon,\bar{\varepsilon}\to 0} \varepsilon^m \bar{\varepsilon}^{\bar{m}} \int d^2y\, y^{j-m-1} \bar{y}^{j-\bar{m}-1} \Phi_j(x+y,\bar{x}+\bar{y};z+\varepsilon,\bar{z}+\bar{\varepsilon}) \Phi_{\frac{k}{2}}(x,\bar{x};z,\bar{z}) \;, \tag{4}$$

where $\Phi_{\frac{k}{2}}$ is the so-called spectral flow operator.

An important additional feature is that the amplitudes in WZW models should satisfy the Knizhnik-Zamolodchikov (KZ) equations [4] which for some N point function A_N in the x-basis read

$$(k-2)\frac{\partial A_N}{\partial z_i} = \sum_{n=1,\, n\neq i}^{N} \frac{1}{z_i - z_n}\left[(x_n - x_i)^2 \frac{\partial^2}{\partial x_i \partial x_n} + 2(x_n - x_i)\left(j_n \frac{\partial}{\partial x_i} - j_i \frac{\partial}{\partial x_n}\right) - 2 j_i j_n\right] A_N \;, \tag{5}$$

(for $1 \leq i \leq N$). In addition, we also point out that, since the spectral flow operator has a null descendant, any correlation function including such an operator at, say, (x_l, z_l) (with $1 \leq l \leq N$) should obey the null vector equation of the form

$$0 = \sum_{n=1,\, n\neq l}^{N} \frac{x_n - x_l}{z_l - z_n}\left[(x_n - x_l)\frac{\partial}{\partial x_n} + 2 j_n\right] A_N \;. \tag{6}$$

In this work we will comment on the generalization of the KZ and null vector equations to the case of amplitudes involving $w = 1$ operators in the x-basis, and on certain three and four point functions including spectral flowed operators, which were computed in [1, 2]. The article is organized as follows. In section 2 we consider the KZ and null vector equations as explained above. In section 3 we focus on three point functions involving $w = 1$ operators whereas in section 4 we consider four point functions in a

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[2] We point out that in the x-basis the operators are labeled with positive w.

somehow more general way, since we deal with operators in arbitrary spectral flow sectors and in both the x- and m-basis.[3] Throughout this work we follow the conventions and notation in [3].

2. KZ AND NULL VECTOR EQUATIONS

In order to obtain the generalizations of (5) and (6) to the case of an N point function including one $w=1$ operator and $N-1$ unflowed operators, the proposal in [1] was to start from an amplitude of $N+1$ unflowed operators, one of which was required to be a spectral flow operator. This should satisfy (5) with the replacement of N by $N+1$. Then, by performing the operation (4) in order to construct ourselves a $w=1$ operator inside the correlator, we should be able to find the generalization of (5) mentioned above. The result in [1] was

$$(k-2)\frac{\partial A_N^w(J)}{\partial z_i} \;=\; -\left(j_1-J+\frac{k}{2}-1\right)\frac{x_2-x_i}{(z_i-z_2)^2}\left[(x_2-x_i)\frac{\partial}{\partial x_i}-2j_i\right]A_N^w(J+1)$$
$$+\frac{1}{z_i-z_2}\left[(x_2-x_i)^2\frac{\partial^2}{\partial x_i\partial x_2}+2(x_2-x_i)\left(J\frac{\partial}{\partial x_i}-j_i\frac{\partial}{\partial x_2}\right)-2j_iJ\right]A_N^w(J) \qquad (7)$$
$$+\sum_{n=3,\,n\neq i}^{N+1}\frac{1}{z_i-z_n}\left[(x_n-x_i)^2\frac{\partial^2}{\partial x_i\partial x_n}+\;2(x_n-x_i)\left(j_n\frac{\partial}{\partial x_i}-j_i\frac{\partial}{\partial x_n}\right)-2j_ij_n\right]A_N^w(J)$$

(for $3\leq i\leq N$). Here J is the spin of the $w=1$ field. Notice that the $w=1$ operator is possitioned at (x_2,z_2) and the possition (x_1,z_1) is no longer present, since it had been occupied by the unflowed operator with spin j_1 which was then fussed with the spectral flow operator at (x_2,z_2). In the expression above, $A_N^w(J)$ stands for the amplitude of one $w=1$ operator and $N-1$ unflowed operators, and the notation $A_N^w(J+1)$ means that we replace $J\longrightarrow J+1$ in A_N^w, so that (7) is actually an iterative expression in the spin of the spectral flowed field. We point out that a similar equation holds for the antiholomorphic part, where $\bar{J}$ turns out to be the iterative variable.

Since A_N^w was obtained out of an $N+1$ point function including one spectral flow operator, then it must also satisfy a generalization of (6). This is obtained through a similar procedure as above giving [1]

$$\left(j_1+J-\frac{k}{2}-1\right)A_N^w(J-1)=\sum_{n=3}^{N+1}\frac{x_n-x_2}{z_2-z_n}\left[(x_n-x_2)\frac{\partial}{\partial x_n}+2j_n\right]A_N^w(J)\,, \qquad (8)$$

which is iterative as well. As before, a similar expression exists for the antiholomorphic part.

In this way, Eqs.(7) and (8), respectively generalize (5) and (6) to the case of an N point function including one $w=1$ operator. The case of an N point function including,

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[3] See [3, 5] for the computation of the two point function for operators in arbitrary spectral flow sectors.

say, M $w=1$ operators and $N-M$ unflowed operators (with $1 \leq M \leq N$) is obtained in a similar manner by starting from an $N+M$ point function including M spectral flow operators. This gives expressions which are iterative in the spins of the M $w=1$ operators in the resulting N point functions.

It would be interesting to establish the connection between the results above and those in [6] regarding the KZ equation for amplitudes of spectral flowed operators, where the calculations were performed in a different basis than here.

3. THREE POINT FUNCTIONS

Here we summarize the already known examples of three point functions involving $w=1$ operators, in the x-basis, before considering the four point functions in the next section. The three point function of one $w=1$ and two $w=0$ operators was computed in [3]. It is given by

$$
\begin{aligned}
\left\langle \Phi^{w=1,j_1}_{J,\bar{J}}(x_1,z_1)\,\Phi_{j_2}(x_2,z_2)\,\Phi_{j_3}(x_3,z_3)\right\rangle &\sim \frac{\Gamma\left(j_1+J-\frac{k}{2}\right)}{\Gamma(1+J-j_2-j_3)}\,\frac{\Gamma\left(j_2+j_3-\bar{J}\right)}{\Gamma\left(1-j_1-\bar{J}+\frac{k}{2}\right)} \\
&\times\ \frac{1}{V_{conf}}B(j_1)C\left(\frac{k}{2}-j_1,j_2,j_3\right)\pi\frac{1}{\gamma\left(j_1+j_2+j_3-\frac{k}{2}\right)} \\
&\times\ \left(x_{21}^{j_3-j_2-J}x_{31}^{j_2-j_3-J}x_{32}^{J-j_2-j_3}\right)\left(z_{21}^{\Delta_3-\Delta_2-\Delta_1^{w=1}}z_{31}^{\Delta_2-\Delta_3-\Delta_1^{w=1}}z_{32}^{\Delta_1^{w=1}-\Delta_2-\Delta_3}\right) \\
&\times\ (\text{antiholomorphic part})\,, \qquad (9)
\end{aligned}
$$

where $B(j)$ and $C(j_1,j_2,j_3)$ are the coefficients corresponding to the amplitudes of two and three unflowed operators respectively. They were first obtained in [7] for the $SL(2,C)/SU(2)$ model and then analytically extended to $SL(2,R)$ in [3] (see this reference for their expressions in our conventions). Here V_{conf} is the volume of the conformal group of S^2 with two point fixed, and $\gamma(x)=\frac{\Gamma(x)}{\Gamma(1-x)}$.

The following step in the literature was to compute the amplitude of two $w=1$ and one $w=0$ operators in the x-basis. Such a calculation was performed in [1]. The starting point was the five point function of three unflowed operators with two spectral flow operators, which we call A_5. The corresponding dependence on the x_i coordinates was given in [8] (see also [3]). Then, the z_i dependent part was computed in [1] thus giving the following complete expression

$$
\begin{aligned}
A_5 \;=\; & B(j_1)B(j_3)C\left(\frac{k}{2}-j_1,j_2,\frac{k}{2}-j_3\right)|z_{12}|^{k}|z_{13}|^{-2j_1}|z_{14}|^{-2j_2}|z_{15}|^{-2j_3} \\
&\times\ |z_{23}|^{-2j_1}|z_{24}|^{-2j_2}|z_{25}|^{-2j_3}|z_{34}|^{2(\Delta_3-\Delta_1-\Delta_2)}|z_{35}|^{2(\Delta_2-\Delta_1-\Delta_3)}|z_{45}|^{2(\Delta_1-\Delta_2-\Delta_3)} \\
&\times\ |x_{12}|^{2(j_1+j_2+j_3-k)}|\mu_1|^{2(j_1-j_2-j_3)}|\mu_2|^{2(j_2-j_1-j_3)}|\mu_3|^{2(j_3-j_1-j_2)}\,, \qquad (10)
\end{aligned}
$$

$$\mu_1 = \frac{x_{14}\,x_{25}}{z_{14}\,z_{25}} - \frac{x_{15}\,x_{24}}{z_{15}\,z_{24}}\,, \quad \mu_2 = \frac{x_{15}\,x_{23}}{z_{15}\,z_{23}} - \frac{x_{13}\,x_{25}}{z_{13}\,z_{25}}\,, \quad \mu_3 = \frac{x_{13}\,x_{24}}{z_{13}\,z_{24}} - \frac{x_{14}\,x_{23}}{z_{14}\,z_{23}}\,. \tag{11}$$

By performing twice the operation in (4) we find the following expression for the three point function of two $w = 1$ and one $w = 0$ operators [1]

$$\begin{aligned}
&\left\langle \Phi^{w=1,j_1}_{J_1,\bar{J}_1}(x_1,z_1)\,\Phi^{w=1,j_2}_{J_2,\bar{J}_2}(x_2,z_2)\,\Phi_{j_3}(x_3,z_3)\right\rangle \sim\; W(j_1,j_2,j_3,m_1,m_2,\bar{m}_1,\bar{m}_2)\\
&\qquad \times\; \frac{1}{V^2_{conf}}\,B(j_1)\,B(j_2)\,C\left(\frac{k}{2}-j_1,\frac{k}{2}-j_2,j_3\right)\\
&\qquad \times\; x_{12}^{j_3-J_1-J_2}\;\bar{x}_{12}^{j_3-\bar{J}_1-\bar{J}_2}\;x_{13}^{J_2-J_1-j_3}\;\bar{x}_{13}^{\bar{J}_2-\bar{J}_1-j_3}\;x_{23}^{J_1-J_2-j_3}\;\bar{x}_{23}^{\bar{J}_1-\bar{J}_2-j_3}\\
&\qquad \times\; z_{12}^{\Delta_3-\Delta_1^{w=1}-\Delta_2^{w=1}}\;\bar{z}_{12}^{\bar{\Delta}_3-\bar{\Delta}_1^{w=1}-\bar{\Delta}_2^{w=1}}\;z_{13}^{\Delta_2^{w=1}-\Delta_1^{w=1}-\Delta_3}\;\bar{z}_{13}^{\bar{\Delta}_2^{w=1}-\bar{\Delta}_1^{w=1}-\Delta_3}\\
&\qquad \times\; z_{23}^{\Delta_1^{w=1}-\Delta_2^{w=1}-\Delta_3}\bar{z}_{23}^{\bar{\Delta}_1^{w=1}-\bar{\Delta}_2^{w=1}-\Delta_3}\,,
\end{aligned} \tag{12}$$

$$\begin{aligned}
W(j_1,j_2,j_3,m_1,m_2,\bar{m}_1,\bar{m}_2) &= \int d^2u\,d^2v\,u^{j_1-m_1-1}\;\bar{u}^{j_1-\bar{m}_1-1}\;v^{j_2-m_2-1}\;\bar{v}^{j_2-\bar{m}_2-1}\\
&\quad \times|1-u|^{2(j_2-j_1-j_3)}\;|1-v|^{2(j_1-j_2-j_3)}\;|u-v|^{2(j_3-j_1-j_2)}\,.
\end{aligned}$$

The explicit form of W is known as such integrals were computed in [9]. As a successful check, the result in (12) was trasformed to the m-basis in [2] using (1), thus giving precisely the expression for the amplitude of three unflowed operators computed in [10], up to the powers of z_{ij}. This is consistent with the claim that the coefficient of all winding conserving amplitudes is the same in the m-basis (for a given number of external states) [3, 6].

4. FOUR POINT FUNCTIONS

Now we follow [2] and consider winding conserving four point functions for operators in arbitrary spectral flow sectors, both in the m- and x-basis. We first recall that the four point functions of unflowed operators were obtained in [3] by analytically extending to $SL(2,R)$ the results in [7]. We have

$$\begin{aligned}
&\left\langle \Phi_{j_1}(x_1,z_1)\;\Phi_{j_2}(x_2,z_2)\;\Phi_{j_3}(x_3,z_3)\;\Phi_{j_4}(x_4,z_4)\right\rangle\\
&\quad = \int dj\,C(j_1,j_2,j)\,B(j)^{-1}\,C(j,j_3,j_4)\;\mathscr{F}(z,x)\;\bar{\mathscr{F}}(\bar{z},\bar{x})\\
&\quad \times\; |x_{43}|^{2(j_1+j_2-j_3-j_4)}\;|x_{42}|^{-4j_2}\;|x_{41}|^{2(j_2+j_3-j_1-j_4)}\;|x_{31}|^{2(j_4-j_1-j_2-j_3)}\\
&\quad \times\; |z_{43}|^{2(\Delta_1+\Delta_2-\Delta_3-\Delta_4)}\;|z_{42}|^{-4\Delta_2}\;|z_{41}|^{2(\Delta_2+\Delta_3-\Delta_1-\Delta_4)}\;|z_{31}|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)}\,,
\end{aligned} \tag{13}$$

where the integral is over $j=\frac{1}{2}+is$ with $s\in\mathbf{R}_{>0}$. Here $\mathscr{F}$ is a function of the cross ratios

$$x=\frac{x_{21}\,x_{43}}{x_{31}\,x_{42}}\;,\qquad z=\frac{z_{21}\,z_{43}}{z_{31}\,z_{42}}\;,$$

and should be computed by requiring (13) to be a solution of (5).

Expanding $\mathscr{F}$ in powers of z as follows

$$\mathscr{F}(z,x)=z^{\Delta_j-\Delta_1-\Delta_2}\,x^{j-j_1-j_2}\sum_{n=0}^{\infty}f_n(x)\,z^n\;, \tag{14}$$

we find f_0 to obey the hypergeometric equation, so that we have the following linearly independent solutions

$${}_2F_1(j-j_1+j_2,j+j_3-j_4,2j;x)\;,$$

or

$$x^{1-2j}\;{}_2F_1(1-j-j_1+j_2,1-j+j_3-j_4,2-2j;x)\;.$$

Taking into account both the holomorphic and antiholomorphic parts, the result in [3] was that the unique monodromy invariant combination is given by

$$\begin{aligned}|\mathscr{F}(z,x)\,|^2\;=\;&|z|^{2(\Delta_j-\Delta_1-\Delta_2)}\,|x|^{2(j-j_1-j_2)}\Bigg\{\,\Big|\,{}_2F_1(j-j_1+j_2,j+j_3-j_4,2j;x)\,\Big|^2\\&+\lambda\,\Big|\,x^{1-2j}{}_2F_1(1-j-j_1+j_2,1-j+j_3-j_4,2-2j;x)\,\Big|^2\Bigg\}+\cdots\end{aligned} \tag{15}$$

where the ellipses denote higher orders in z and

$$\lambda=-\,\frac{\gamma(2j)^2\,\gamma(-j_1+j_2-j+1)\,\gamma(j_3-j_4-j+1)}{(1-2j)^2\,\gamma(-j_1+j_2+j)\,\gamma(j_3-j_4+j)}\;.$$

It has been found in [7] that the higher orders in (14) are determined iteratively by the KZ equation starting from f_0 as the initial condition.

Now we would like to extend the results above to the case of winding conserving four point functions for states in arbitrary spectral flow sectors, both in the *m*- and *x*-basis. The idea is to first employ (1) in order to transform (13) to the *m*-basis where we could exploit the fact that, as mentioned and successfully checked in the previous section, the coefficient of all winding conserving amplitudes is the same in the *m*-basis (for a given number of external states). After performing the spectral flow, we should transform the final result back to the *x*-basis. In order to simplify the calculations, the only requirement that we will make is that at least one state is in the spectral flow image of the highest weight discrete representation. After heavy calculations, it was found in [2] that the winding conserving four point functions for states in arbitrary spectral flow

sectors, in the m-basis, read

$$
\begin{aligned}
&\left\langle \Phi^{w_1, j_1=-m_1-n_1=-\bar{m}_1-\bar{n}_1}_{J_1,M_1;\bar{J}_1,\bar{M}_1}(z_1)\, \Phi^{w_2,j_2}_{J_2,M_2;\bar{J}_2,\bar{M}_2}(z_2)\, \Phi^{w_3,j_3}_{J_3,M_3;\bar{J}_3,\bar{M}_3}(z_3)\, \Phi^{w_4,j_4}_{J_4,M_4;\bar{J}_4,\bar{M}_4}(z_4) \right\rangle \\
&\sim V_{conf}\; \delta^2((m_1-n_1)+m_2+m_3+m_4)\, \frac{\Gamma(2j_1)^2}{\Gamma(j_1-m_1)\,\Gamma(j_1-\bar{m}_1)} \\
&\times\; z_{43}^{\Delta_1^{w_1}(n_1)+\Delta_2^{w_2}-\Delta_3^{w_3}-\Delta_4^{w_4}}\; \bar{z}_{43}^{\bar{\Delta}_1^{w_1}(\bar{n}_1)+\bar{\Delta}_2^{w_2}-\bar{\Delta}_3^{w_3}-\bar{\Delta}_4^{w_4}}\; z_{42}^{-2\Delta_2^{w_2}}\; \bar{z}_{42}^{-2\bar{\Delta}_2^{w_2}} \\
&\times\; z_{41}^{\Delta_2^{w_2}+\Delta_3^{w_3}-\Delta_1^{w_1}(n_1)-\Delta_4^{w_4}}\; \bar{z}_{41}^{\bar{\Delta}_2^{w_2}+\bar{\Delta}_3^{w_3}-\bar{\Delta}_1^{w_1}(\bar{n}_1)-\bar{\Delta}_4^{w_4}} \\
&\times\; z_{31}^{\Delta_4^{w_4}-\Delta_1^{w_1}(n_1)-\Delta_2^{w_2}-\Delta_3^{w_3}}\; \bar{z}_{31}^{\bar{\Delta}_4^{w_4}-\bar{\Delta}_1^{w_1}(\bar{n}_1)-\bar{\Delta}_2^{w_2}-\bar{\Delta}_3^{w_3}} \\
&\times \sum_{n_2,n_3=0}^{n_1} \sum_{\bar{n}_2,\bar{n}_3=0}^{\bar{n}_1} \mathscr{G}_{n_2,n_3}(j_i,m_i)\, \mathscr{G}_{\bar{n}_2,\bar{n}_3}(j_i,\bar{m}_i) \int dj\, C(j_1,j_2,j)\, B(j)^{-1} \\
&\qquad \times\; C(j,j_3,j_4) \Big[\Omega(j,j_i,m_2-n_2,m_3-n_3,\bar{m}_2-\bar{n}_2,\bar{m}_3-\bar{n}_3) \\
&\qquad\qquad + \lambda\, \Omega(1-j,j_i,m_2-n_2,m_3-n_3,\bar{m}_2-\bar{n}_2,\bar{m}_3-\bar{n}_3) \Big] \\
&\qquad \times\; z^{\Delta_j^{w}-\Delta_1^{w_1}(n_1)-\Delta_2^{w_2}}\; \bar{z}^{\bar{\Delta}_j^{w}-\bar{\Delta}_1^{w_1}(\bar{n}_1)-\bar{\Delta}_2^{w_2}} + \cdots ,
\end{aligned} \tag{16}
$$

where the ellipses denote higher orders in z. We require that this is a winding conserving correlator, *i.e.* we should have $\sum_{i=1}^{4} w_i = 0$. The operator at z_1 is assumed to be in the spectral flow image of the highest weight discrete representation, so that we have introduced the number $n_1 = 0,1,2,\cdots$, which satisfies $m_1 = -j_1 - n_1$. We also have

$$
\begin{aligned}
\mathscr{G}_{n_2,n_3}(j_i,m_i) \equiv{}& \frac{1}{\Gamma(n_2+1)\Gamma(n_3+1)}\, \frac{\Gamma(-j_1-m_1+1)}{\Gamma(-j_1-m_1-n_2-n_3+1)} \\
&\times\; \frac{\Gamma(j_2-m_2+n_2)\Gamma(j_3-m_3+n_3)\Gamma(j_4-j_1-m_4-m_1-n_2-n_3)}{\Gamma(j_2-m_2)\Gamma(j_3-m_3)\Gamma(j_4-m_4)} ,
\end{aligned}
$$

and

$$
\begin{aligned}
\Omega(j,j_i,m_i,\bar{m}_i) ={}& \Gamma(-j_3-j_4+j+1)^2\, \Gamma(j_3-m_3)\, \Gamma(j_3-\bar{m}_3) \\
\times \Bigg(& \Lambda \left[\begin{matrix} -j_1+j_2+j,\ -j_1+j_2-j+1,\ -j_1+j_2+j_4+m_3,\ 1 \\ -j_1+j_2-j_3+j_4+1,\ -j_1+j_2+j_3+j_4,\ j_2-m_2+1 \end{matrix} \right] \\
+ \Lambda & \left[\begin{matrix} -j_3-j_4+j+1,\ -j_3-j_4-j+2,\ -j_3+m_3+1,\ j_1-j_2-j_3-j_4+2 \\ j_1-j_2-j_3-j_4+2,\ -2j_3+2,\ j_1-j_3-j_4-m_2+2 \end{matrix} \right]
\end{aligned}
$$

$$
+\,\Lambda\left[\begin{array}{c} j_3-j_4+j,\ j_3-j_4-j+1,\ j_3+m_3,\ j_1-j_2+j_3-j_4+1 \\ j_1-j_2+j_3-j_4+1,\ 2j_3,\ j_1+j_3-j_4-m_2+1 \end{array}\right]\Bigg)
$$

$$
\times\ \Bigg(\Lambda\left[\begin{array}{c} -j_1+j_2+j,\ -j_1+j_2-j+1,\ -j_1+j_2+j_4+\bar{m}_3,\ 1 \\ -j_1+j_2-j_3+j_4+1,\ -j_1+j_2+j_3+j_4,\ j_2-\bar{m}_2+1 \end{array}\right]
$$

$$
+\,\Lambda\left[\begin{array}{c} -j_3-j_4+j+1,\ -j_3-j_4-j+2,\ -j_3+\bar{m}_3+1,\ j_1-j_2-j_3-j_4+2 \\ j_1-j_2-j_3-j_4+2,\ -2j_3+2,\ j_1-j_3-j_4-\bar{m}_2+2 \end{array}\right]
$$

$$
+\,\Lambda\left[\begin{array}{c} j_3-j_4+j,\ j_3-j_4-j+1,\ j_3+\bar{m}_3,\ j_1-j_2+j_3-j_4+1 \\ j_1-j_2+j_3-j_4+1,\ 2j_3,\ j_1+j_3-j_4-\bar{m}_2+1 \end{array}\right]\Bigg)\,, \tag{17}
$$

with

$$
\Lambda\left[\begin{array}{c} \rho_1,\rho_2,\rho_3,\rho_4 \\ \sigma_1,\sigma_2,\sigma_3 \end{array}\right] \equiv \frac{\Gamma(1-\sigma_1)\,\Gamma(1-\sigma_2)\,\Gamma(1+\rho_1-\rho_2)\,\Gamma(\rho_4)\,\Gamma(\sigma_3-\rho_4)}{\Gamma(1-\rho_2)\,\Gamma(1-\rho_3)\,\Gamma(1+\rho_1-\sigma_1)\,\Gamma(1+\rho_1-\sigma_2)\,\Gamma(\sigma_3)}
$$

$$
\times\ (-1)^{\rho_1}\,{}_4F_3(\rho_1,\rho_2,\rho_3,\rho_4;\sigma_1,\sigma_2,\sigma_3;1)\,.
$$

We complete our analysis by transforming (16) back to the x-basis. We take into account that the x-basis correlators are the pole residues of the m-basis results at $J_i = M_i$, $\bar{J}_i = \bar{M}_i$, for a given spectral flowed state. In addition, we point out that the Ward identities satisfied by correlators involving either unflowed or spectral flowed fields in the x-basis are the same up to the replacements $\Delta_i \longrightarrow \Delta_i^w,\ j_i \longrightarrow J_i$ for the spectral flowed operators [1], and further, that, to the lowest order in z, the modified KZ equation (7) actually reduces to that of the unflowed case with the replacements $j_i \longrightarrow J_i$, and the iterative terms do not contribute.[4] Taking all this into account, we find the following expression in the x-basis [2]

$$
\left\langle \Phi^{|w_1|,j_1}_{J_1(n_1),\bar{J}_1(\bar{n}_1)}(x_1,z_1)\,\Phi^{|w_2|,j_2}_{J_2,\bar{J}_2}(x_2,z_2)\,\Phi^{|w_3|,j_3}_{J_3,\bar{J}_3}(x_3,z_3)\,\Phi^{|w_4|,j_4}_{J_4,\bar{J}_4}(x_4,z_4)\right\rangle
$$

$$
\sim\ \frac{\Gamma(2j_1)^2}{\Gamma(j_1-m_1)\,\Gamma(j_1-\bar{m}_1)}\,x_{43}^{J_1(n_1)+J_2-J_3-J_4}\ \bar{x}_{43}^{\bar{J}_1(\bar{n}_1)+\bar{J}_2-\bar{J}_3-\bar{J}_4}\ x_{42}^{-2J_2}\ \bar{x}_{42}^{-2\bar{J}_2}
$$

$$
\times\ x_{41}^{J_2+J_3-J_1(n_1)-J_4}\ \bar{x}_{41}^{\bar{J}_2+\bar{J}_3-\bar{J}_1(\bar{n}_1)-\bar{J}_4}\ x_{31}^{J_4-J_1(n_1)-J_2-J_3}\ \bar{x}_{31}^{\bar{J}_4-\bar{J}_1(\bar{n}_1)-\bar{J}_2-\bar{J}_3}
$$

$$
\times\ z_{43}^{\Delta_1^{|w_1|}(n_1)+\Delta_2^{|w_2|}-\Delta_3^{|w_3|}-\Delta_4^{|w_4|}}\ \bar{z}_{43}^{\bar{\Delta}_1^{|w_1|}(\bar{n}_1)+\bar{\Delta}_2^{|w_2|}-\bar{\Delta}_3^{|w_3|}-\bar{\Delta}_4^{|w_4|}}\ z_{42}^{-2\Delta_2^{|w_2|}}\ \bar{z}_{42}^{-2\bar{\Delta}_2^{|w_2|}}
$$

$$
\times\ z_{41}^{\Delta_2^{|w_2|}+\Delta_3^{|w_3|}-\Delta_1^{|w_1|}(n_1)-\Delta_4^{|w_4|}}\ \bar{z}_{41}^{\bar{\Delta}_2^{|w_2|}+\bar{\Delta}_3^{|w_3|}-\bar{\Delta}_1^{|w_1|}(\bar{n}_1)-\bar{\Delta}_4^{|w_4|}}
$$

---

[4] Such a property can be shown to hold for correlators involving $w=1$ operators, as in (7). However it seems reasonable to assume that this can be extended to arbitrary winding number.

$$
\begin{aligned}
&\times\ z_{31}^{\Delta_4^{|w_4|}-\Delta_1^{|w_1|}(n_1)-\Delta_2^{|w_2|}-\Delta_3^{|w_3|}}\ \bar{z}_{31}^{\bar{\Delta}_4^{|w_4|}-\bar{\Delta}_1^{|w_1|}(\bar{n}_1)-\bar{\Delta}_2^{|w_2|}-\bar{\Delta}_3^{|w_3|}} \\
&\times\ \sum_{n_2,n_3=0}^{n_1}\ \sum_{\bar{n}_2,\bar{n}_3=0}^{\bar{n}_1} \mathscr{G}_{n_2,n_3}(j_i,m_i)\,\mathscr{G}_{\bar{n}_2,\bar{n}_3}(j_i,\bar{m}_i)\int dj\, C(j_1,j_2,j)\, B(j)^{-1} \\
&\qquad \times\ C(j,j_3,j_4)\,\Big[\Omega(j,j_i,m_2-n_2,m_3-n_3,\bar{m}_2-\bar{n}_2,\bar{m}_3-\bar{n}_3) \\
&\qquad\qquad + \lambda\,\Omega(1-j,j_i,m_2-n_2,m_3-n_3,\bar{m}_2-\bar{n}_2,\bar{m}_3-\bar{n}_3)\Big] \\
&\qquad \times\ z^{\Delta_j^{|w|}-\Delta_1^{|w_1|}(n_1)-\Delta_2^{|w_2|}}\ \bar{z}^{\bar{\Delta}_j^{|w|}-\bar{\Delta}_1^{|w_1|}(\bar{n}_1)-\bar{\Delta}_2^{|w_2|}}\ x^{j-J_1(n_1)-J_2}\ \bar{x}^{j-\bar{J}_1(\bar{n}_1)-\bar{J}_2} \\
&\qquad \times\ \Big\{\hat{\lambda}(n_1)\left|x^{1-2j}\ {}_2F_1(1-j-J_1(n_1)+J_2,1-j+J_3-J_4,2-2j;x)\right|^2 \\
&\qquad\qquad + \left|{}_2F_1(j-J_1(n_1)+J_2,j+J_3-J_4,2j;x)\right|^2\Big\} + \cdots\,,
\end{aligned}
\tag{18}
$$

where the ellipses denote higher order terms in z, and we have $J_1(n_1) = |-j_1 - n_1 + \frac{k}{2}w_1|$. Notice that we have replaced $w_i \to |w_i|$ due to the fact that in the x-basis the operators are labeled with positive winding number. We also have

$$
\hat{\lambda} = -\frac{\gamma(2j)^2\,\gamma(-J_1+J_2-j+1)\,\gamma(J_3-J_4-j+1)}{(1-2j)^2\,\gamma(-J_1+J_2+j)\,\gamma(J_3-J_4+j)}\,. \tag{19}
$$

A comment is in order. We emphasize that, whereas the m-basis expression (16) holds for all winding conserving four point functions, including as a particular case the situation in which all the external operators are unflowed, (18) does not hold when all the external states are unflowed. This is consistent with the fact that in the m-basis, all N point functions are the same, for a given N, up to some free boson correlator which only modifies the z_i dependence. We point out that the calculations we have performed in order to transform the amplitude from the m- to the x-basis, involving the evaluation of the pole residue at $J = M$, $\bar{J} = \bar{M}$, require at least one spectral flowed state in the correlator. Thus (18) results in this case, whereas (13) holds for four unflowed operators.

We also emphasize that the higher order contributions to the expansions in z are iteratively determined using modified KZ equations, such as (7), for amplitudes involving spectral flowed states, starting from (18) as the initial condition.

Finally, we point out that one important application of our results would be to investigate the structure of the factorization of (18) in order to establish the consistency of string theory on AdS_3.

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